“Drawing Illusions” – a case study in the incorrectness of diagrammatic reasoning

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Abstract

In "Something to Reckon With" [6], a system for diagramming syllogistic inferences using straight line segments is presented (see also Englebretsen [5]). In the light of recent research on the representational power of diagrammatic representation systems (Lemon and Pratt [12, 13]) we point out some problems with the proposal, and indeed, with any proposal for representing logically possible situations diagrammatically. We shall first outline the proposed linear diagrammatic system of Englebretsen [5], and then show by means of counterexamples that it is inadequate as a representation scheme for general logical inferences (the task for which the system is intended). We also show that modifications to the system fail to remedy the problems. The considerations we present are not limited to the particular proposals of Englebretsen [5, 6]; we thus draw a more general moral about the use of spatial relations in representation systems.

1 Diagrammatic representation systems

Diagrammatic representation systems are of increasing interest for at least two reasons. Philosophically, diagram systems interest those concerned with the nature of representation itself – in particular, those who argue that too much attention has been given to sequential symbol systems. These writers claim that diagrams represent by "analogy" or "surrogacy" – in virtue of sharing structure with the domains that they represent (eg: Barwise and Shimojima [2], Cummins [4], Swoyer [19]). Practically, diagrammatic representations are frequently used in visual interfaces to databases, programming languages, and in logic teaching. Each of these domains demands careful consideration of the formal properties of the diagrammatic systems in question. For both of these reasons we propose to investigate the expressive power of one proposed diagram system, and to determine its utility in reasoning tasks.

2 The diagrammatic system LD

In [5, 6], George Englebretsen presents us with a system for diagramming syllogistic inferences. In this system, individuals are represented by labelled points, and sets are represented by labelled straight line segments. Relationships between sets and individuals or between sets and sets are then represented by incidence relations between the corresponding points and line segments. The author explains how such a representation system can be used to carry out syllogistic reasoning tasks, and illustrates his explanation with many examples. We shall call the basic system of Englebretsen [5] (i.e. without representation of relations or pronouns) "LD" for "linear diagrams". Figure 1 shows an example of a diagram of LD, together with its intended interpretation.

More formally, we may reconstruct LD as follows. A linear diagram is a finite set of "dots" and "dashes". A dot is simply a labelled point in the plane, and a dash is a finite line segment with a dot attached to its right terminus. The restriction to right termini is Englebretsen's, not ours: Englebretsen does

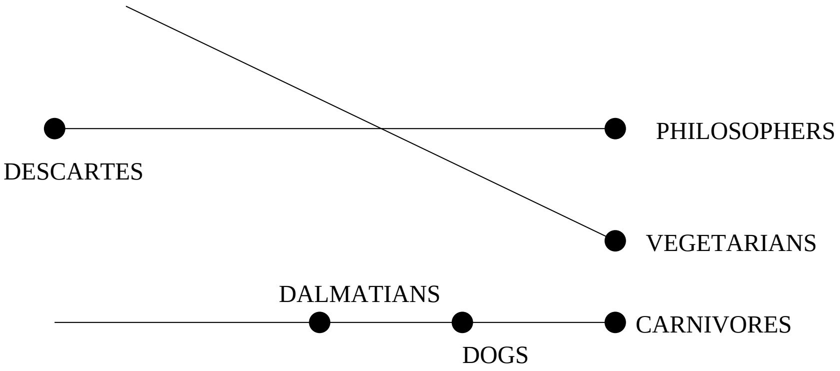

Figure 1: Some philosophers are vegetarians, Descartes is a philosopher, no dogs are philosophers, all dalmatians are dogs, all dogs are carnivores.

not consider vertical dashes, but we may assume that these are labelled by dots at their upper terminus. The terminology of “dots” and “dashes” is ours, not Englebretsen’s: it is introduced simply to avoid confusion with the infinitely many unmarked points and lines in the space occupied by an LD diagram, and changes nothing of substance.

LD is to be interpreted as follows.

Definition 1 *(Linear diagrams, Englebretsen [5] p. 38-47)*

1. *Dashes represent sets.*
2. *Any dot which is not the right terminus of some dash represents an individual.*
3. *The presence of a dot on a dash (other than its right terminus) indicates that the individual represented by the dot is a member of the set represented by the dash.*
4. *That two dashes intersect represents the fact that the sets they represent have a nonempty intersection.*
5. *That two dashes do not intersect represents the fact that the sets they represent have an empty intersection.*

6. *That a dash l lies within a dash l' represents the fact that the set represented by l is a subset of the set represented by l'.*

Actually, Englebretsen further insists that dashes representing a set L and not-L (the complement of L) must be parallel. However, given that line segments, rather than lines, are employed in the system it is unclear precisely what is gained by insisting that line segments representing complementary sets be parallel (as opposed to merely non-intersecting). On the other hand, it is not clear that any harm is done either.

Note that there is some ambiguity regarding the representation of individuals as dots. It is not clear from [5] whether two coincident dots must represent the same individual; nor is it clear whether one individual can be represented by multiple (in particular, non-coincident) dots. Englebretsen claims that,

> "... identity statements are ... easily diagrammed by our method. A proposition of the form 'a is (identical to) b' ... means that 'a' and 'b' label the same point." ([5], p. 47).

But it is unclear how to interpret this pronouncement. Certainly, Englebretsen cannot wish to claim that every *point* in the plane can represent no more than one individual. For "some As are Bs" is to be represented by line intersection, and that intersection, though a single point, may contain more than one individual. However, in the counterexamples which we present below, we take no particular stance as regards these issues.

Inference in LD is to be carried out, as is usual with diagrammatic representations, by enumeration of cases. That is, a conclusion follows from a set of premises if all ways of diagramming the premises result in a diagram depicting the conclusion. Clearly, it follows from the premises of figure 1 (some philosophers are vegetarians, Descartes is a philosopher, no dogs are philosophers, all dalmatians are dogs, all dogs are carnivores) that all dalmatians are carnivores, since this fact must hold (by part 6 of definition 1) however, exactly, the diagram of the premises is drawn. Equally clearly, it does not follow that

Descartes is a carnivore, because the diagram shows us a way for the premises to be true and the conclusion false.

While there has been much recent research in the logical analysis of diagrammatic reasoning (Allwein and Barwise [1], Barwise and Shimojima [2], Glasgow et al. [8], Shimojima [14], Shin [15], Stenning and Oberlander [18]) we believe that some basic properties of diagrammatic representation systems deserve more attention, particularly when their suitability for performing logical inferences is in question. In particular, the following two properties of representation systems are of central importance.

1. For every representation of the system there is some possible situation of which it is true ("self-consistency").

2. Every possible situation has some representation true of it.

The inference system of Euler's Circles (see e.g. Hammer [10]) has been shown (Lemon and Pratt [13]) to exhibit the first, but not the second of these properties, thus making it unsuitable for syllogistic inference in general. In particular, since not all logical possibilities can be represented in the system, attempts to diagram some situations will lead to incorrect inferences. In the present paper we perform a similar analysis for the proposal to use linear diagrams in syllogistic inference.

Englebretsen's stated aim is to design a diagram system which is simple but which also avoids the expressive limitations imposed by the geometry of closed plane figures (e.g. Euler's Circles) – limitations which, according to Englebretsen, are often overlooked by those in the "diagrammatic reasoning" community. Thus:

> "The geometric restrictions on closed plane figures which prevent *perspicuous* representations involving more than four terms using simple continuous figures do not apply to the still simpler linear figures." (Englebretsen [5], p. 47, our italics)

" . . . the major advantage of line diagrams is their ability to represent inferences involving relatively large numbers of terms . . ." (Englebretsen [5], p. 46)

We are not told which geometric restrictions Englebretsen has in mind (the above claims are not proven formally); he merely mentions a four-term limit on the use of closed plane figures, due to geometric restrictions from the use of the plane as a representational medium. The reference given (Gardner [7]) mentions only a "virtual four term limit" based on psychological rather than geometrical restrictions, which suggests that the restrictions Englebretsen seeks to avoid are practical in nature. Roughly, the problem seems to be that, when diagrammatic systems are used to represent large numbers of premises, the result tends to look like a plate of spaghetti.

We shall show in sections 3 and 4 that, contrary to Englebretsen's assertions, the system LD is subject to geometrical constraints which compromise its utility for logical inference, regardless of considerations of perspicuity and readability. In earlier work (Lemon and Pratt [12, 13]), it has been demonstrated that other diagrammatic representation systems, based on the representation of sets by areas, fall victim to similar problems. Thus, Englebretsen's claim to have overcome the expressive limitations imposed by plane geometry through the use of line segments rather than areas cannot be maintained.

3 First counterexample to the correctness of LD

Let P_i $(1 \leq i \leq 3)$ be individuals and L_j $(1 \leq j \leq 3)$ sets. Consider the following situation:

> Individual P_i is a member of the set L_j if and only if $i \neq j$ $(1 \leq i \leq 3,$ $1 \leq j \leq 3)$.

It is clear that this situation corresponds to a finite set of statements of the forms "P is an L" and "P is not an L", and thus falls within the purview of

LD. Indeed, it can be diagrammed as shown in figure 2. Here, the individuals P_i are represented by dots p_i ($1 \le i \le 3$), and the sets l_i are represented by the dashes l_i ($1 \le i \le 3$).

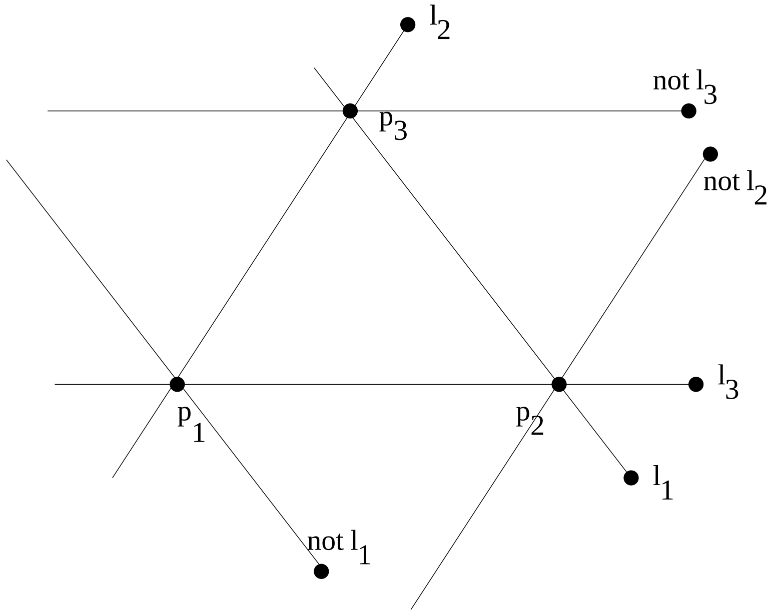

Figure 2: A linear diagram.

But now let P_4 be a fourth individual, and consider above the situation but augmented by:

Individual P_4 is a member of the all the sets L_j ($1 \le j \le 3$).

To see why this augmented situation cannot be diagrammed in LD, suppose that the new individual is represented by a dot p_4. Then the four dots p_i must all be distinct (since no two may lie on exactly the same dashes l_j). It follows that the dashes l_1 and l_2 are collinear, since they both contain the distinct dots p_3 and p_4; by similar reasoning, l_1 and l_3 must be collinear, so all the dashes lie on some common line λ (say).

Orient the diagram so that λ is horizontal. Let us write $p \prec p'$ to indicate that dot p is to the left of dot p' (with the obvious interpretation for

$p \preceq p'$). Finally, let $\dot{l}_j$ stand for the right terminus of the dash l_j ($1 \leq j \leq 3$). It follows from what we are told about L_1 that either $p_2, p_3, p_4 \preceq \dot{l}_1 \prec p_1$ or $p_1 \prec p_2, p_3, p_4 \preceq \dot{l}_1$. Whence, from what we are told about L_2, either $p_2 \prec p_3, p_4 \preceq \dot{l}_1 \prec p_1$ or $p_1 \prec p_3, p_4 \prec p_2 \preceq \dot{l}_1$. Either way, p_3 lies between p_1 and p_2, which contradicts what we are told about L_3.

Note that this type of behaviour means that LD could be used to make invalid inferences. For instance, suppose we omit from the above (augmented) situation the fact that individual P_1 does not belong to set L_1. Then the above geometrical argument shows that all ways of diagramming the remaining facts will force the dot p_1 to lie on the dash l_1, thus inviting the inference that P_1 belongs to L_1. Of course, this inference would be invalid.

There is a way in which Englebretsen might save his system from the problem just raised. He could point out that the restriction to *straight* line segments is inessential. To be sure, the original motivation for using lines to represent sets was to avoid the unreadable diagrams resulting from region-based representation systems; but perhaps there is a middle way. For example, one might represent sets as connected chains of straight line segments, or even as arbitrary algebraic curves. Since we can only guess at the possibilities here, we shall assume only that the plane figures used to represent sets are algebraic curves. This generalization of Englebretsen's system, which we call "curved LD", will now be investigated.

4 Second counterexample

We now investigate "curved LD"—the generalization of LD employing algebraic curves instead of straight line segments. Henceforth, then, we use the term "dash" to refer to algebraic curves (continuous algebraic functions from $[0,1]$ to $\mathbb{R}^2$) in a diagram, with dots at one of their endpoints, where, again, we interpret dashes as representing sets.

Let P_i ($1 \leq i \leq 5$) be individuals and L_{jk} ($1 \leq j < k \leq 5$) sets. Consider the following situation (we label it S):

Individual P_i is a member of the set L_{jk} if and only if $i = j$ or $i = k$.

No L_{jk}s are $L_{j'k'}$s if $\{j, k\} \cap \{j', k'\} = \emptyset$.

It is clear that this situation corresponds to a finite set of statements of the forms "P is an L", "P is not an L", and "No Ls are L's", and thus falls within the purview of LD; let us see how we might represent it.

Let each individual P_i be represented by a dot p_i ($1 \leq i \leq 5$) and each set L_{ij} by a dash l_{ij} ($1 \leq i < j \leq 5$). Then the five dots p_i must all be distinct (since no two may lie on exactly the same dashes l_{kj}). Now, each dash l_{ij} ($1 \leq i < j \leq 5$) contains the dots p_i and p_j, so that every pair of the five points is to be joined by some dash (an algebraic curve). In addition, the algebraic curves l_{jk} and $l_{j'k'}$ may not intersect if they do not share one of the p_i.

Now consider the following definition and theorem – a generalization of the well-known non-planarity result for the graph K_5 (see Bollobás [3]).

Definition 2 *Let $v_1, \ldots, v_5$ be distinct points in the plane and let $\{v_i \to v_j\}_{1 \leq i < j \leq 5}$ be algebraic curves such that $v_i \to v_j(0) = v_i$ and $v_i \to v_j(1) = v_j$. We say that the $\{v_i \to v_j\}_{1 \leq i < j \leq 5}$ are* drawn without intersection violations *if $v_i \to v_j$ and $v_{i'} \to v_{j'}$ are disjoint for $\{i, j\}$ and $\{i', j'\}$ disjoint.*

Theorem 1 *The $\{v_i \to v_j\}_{1 \leq i < j \leq 5}$ cannot be drawn without intersection violations.*

Proof:

(We conflate the functions $\{v_i \to v_j\}_{1 \leq i < j \leq 5}$ with their ranges throughout). If $v_1 \to v_2$ and $v_1 \to v_3$ divide the plane into more than one residual domain, it is easy to see that there must be such a residual domain E of $v_1 \to v_2$ and $v_1 \to v_3$ containing none of v_1, v_2, v_3 and such that v_2 does not lie on the boundary of E (since v_2 cannot lie on $v_1 \to v_3$). Suppose that some node v_4 lies in E. Then since E is bounded by $v_1 \to v_2$ and $v_1 \to v_3$, v_5 lies in E, for otherwise $v_4 \to v_5$ could not be drawn without an intersection violation.

Now $v_3 \to v_4$ cannot cross $v_1 \to v_2$, but certainly touches the segment of $v_1 \to v_3$ between v_1 and X in figure 3 – i.e. the segment of $v_1 \to v_3$ bounding E. Let ϵ be the final section of $v_1 \to v_3$ after X, let ζ be the initial section of

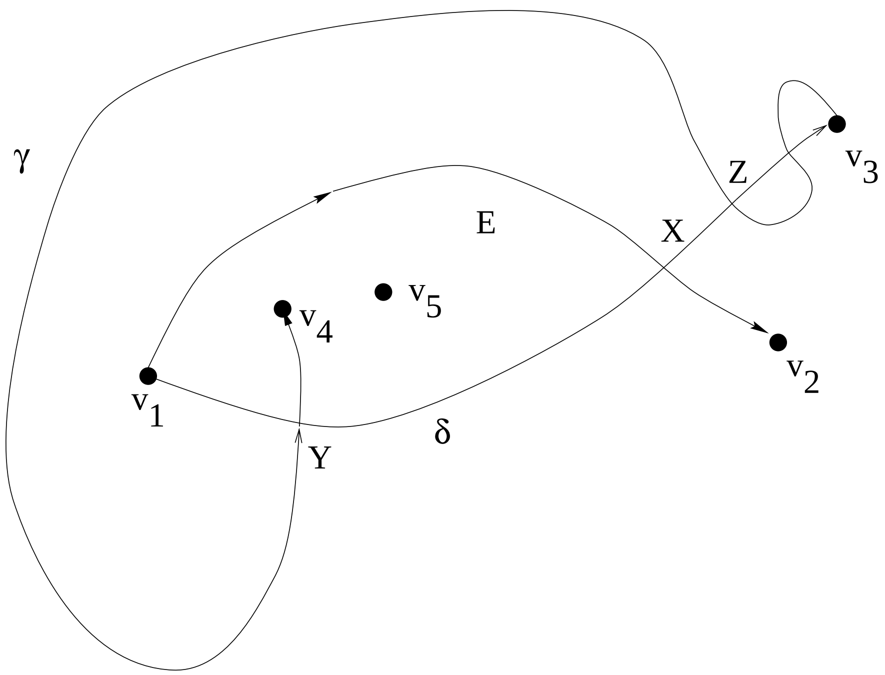

Figure 3:

$v_1 \to v_3$ up to X, let Z be the last point on $v_3 \to v_4$ touching ϵ, let Y be the first point on $v_3 \to v_4$ touching ζ, let γ be the segment of $v_3 \to v_4$ from Z to Y, and let δ be the segment of $v_1 \to v_3$ from Y to Z. Then γ and δ together bound a Jordan region separating v_2 and v_5 so that $v_2 \to v_5$ cannot be drawn. Hence we can choose one such domain E, with no nodes inside E. Suppose E is penetrated by some arc α. Since no nodes lie in E, figures 4 and 5 show the only possibilities for $\alpha : \alpha$ intersects just one of $v_1 \to v_2$ and $v_1 \to v_3$, or α intersects both of them.

If α intersects just one of the two arcs, say $v_1 \to v_2$, then we can replace $v_1 \to v_3$ by α and obtain an exactly similar arrangment involving a smaller residual domain E', as shown in figure 4. Moreover, E' will be penetrated by fewer arcs than E. So we may assume this case does not arise.

Thus, without loss of generality, all arcs penetrating E intersect both $v_1 \to$

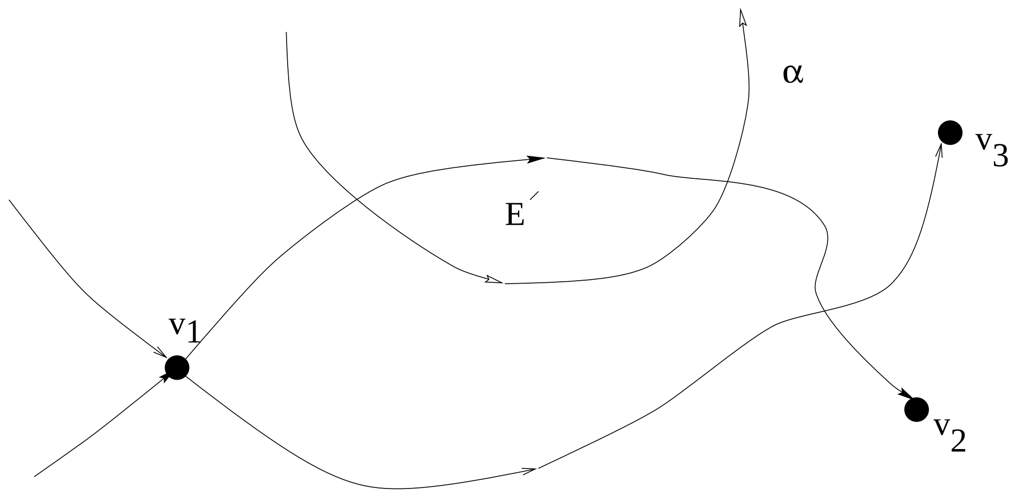

Figure 4:

v_2 and $v_1 \to v_3$. So, let X and Y be consecutive crossing points on $v_1 \to v_2$ as shown in figure 6.

Now switch the segments of $v_1 \to v_2$ and $v_1 \to v_3$ between X and Y. By assumption, all arcs penetrating E cut both $v_1 \to v_2$ and $v_1 \to v_3$, so no intersection violations occur. The result is shown in figure 7.

Let a, b be edges in the abstract graph K_5 not involving any common vertices. Let e be some embedding of K_5 in the plane (allowing some arcs to cross) such that the corresponding arcs $e(a)$ and $e(b)$ do not intersect. Then, because $e(a)$ and $e(b)$ are closed sets, there is a minimum distance $\delta > 0$ between any point on $e(a)$ and any point on $e(b)$. Now let e' be any embedding in which a and b correspond to the arcs $e'(a)$ and $e'(b)$, such that every point in $e'(a)$ lies within $\delta/2$ of some point in $e(a)$ and every point in $e'(b)$ lies within $\delta/2$ of some point in $e(b)$. In other words, e' is a "small adjustment" of e. Then by the triangle inequality for distances, there cannot be a common point of $e'(a)$ and $e'(b)$. Thus, these arcs still do not intersect. A virtually identical argument shows that a sufficiently small adjustment (in the above sense) to any arcs cannot introduce new intersection violations anywhere in the embedding.

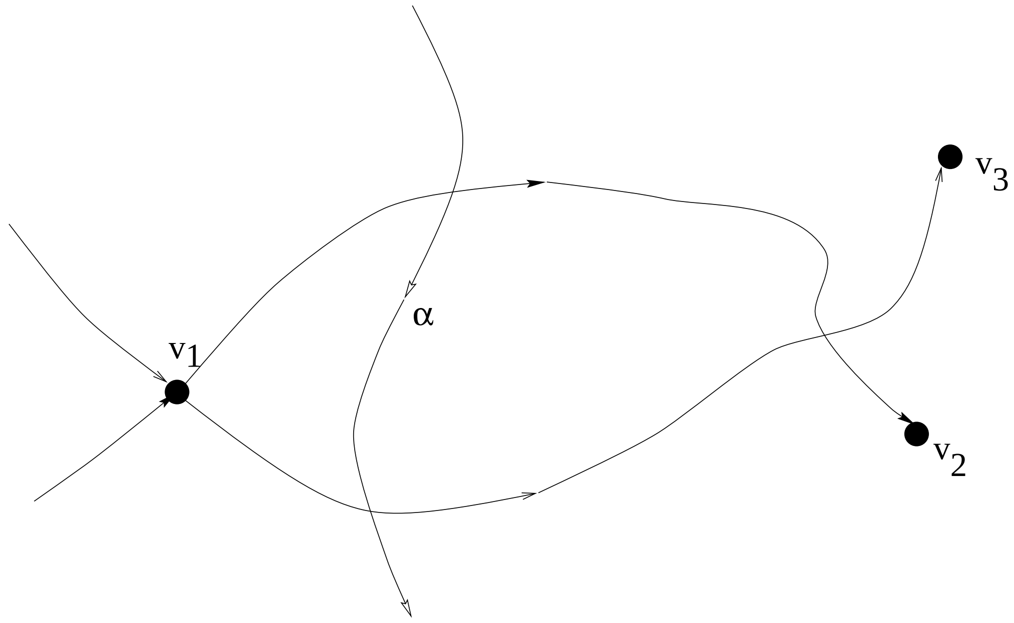

Figure 5:

Now, there exists a small adjustment (in the above sense) which will remove the points of contact X and Y. Thus the number of critical points (points where arcs cross over, touch, merge or split[1]) will have been reduced.

By carrying out successive simplifications of this form, we can be assured that $v_1 \to v_2$ and $v_1 \to v_3$ do not separate the plane into more than one residual domain. Proceeding in this fashion for all triples, all arcs involving a common node must from a tree as shown in figure 8.

By a series of arbitrarily small adjustments, this tree can be converted into a fan (see figure 9) without introducing any intersection violations.

Thus we obtain a planar embedding of the graph K_5, which is impossible.

□

[1]It is obvious from the assumption that the arcs are algebraic that there are at most finitely many such points.

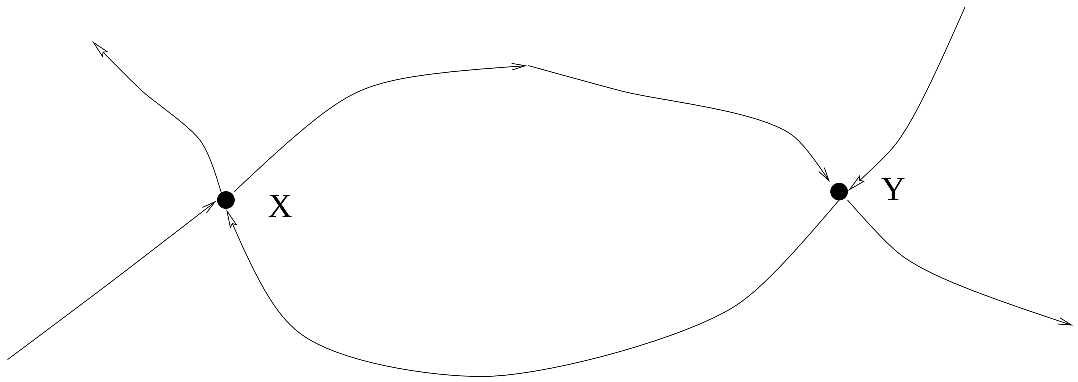

Figure 6:

Returning to our counterexample, and the situation S, note that in any representation of the situation S, the dots p_i ($1 \leq i \leq 5$) and the sections of the dashes l_{ij} lying between p_i and p_j ($1 \leq i < j \leq 5$) constitute drawing (i.e. a plane embedding) of $\{p_i \to p_j\}_{1 \leq i < j \leq 5}$. Since (by theorem 1) these curves cannot be drawn without intersection violations, the constraint that "No L_{jk}s are $L_{j'k'}$s if $\{j,k\} \cap \{j',k'\} = \emptyset$" cannot be met by any representation of the situation S in curved LD. It follows that, for any of the statements in the above situation, its negation will, according to the LD inference procedure, be implied by the others. Of course, such an inference would be invalid. Figure 10 shows one attempt to realize these premises in the system of curved LD. Any other attempt would fail similarly. Again, we have shown that there are consistent statements of set theory which cannot be represented in the system LD.

5 Conclusion

We have shown that the proposed system of Englebretsen [5, 6] for diagramming syllogistic inferences does not manage, by employing lines rather than regions, to avoid important geometric limitations on plane figures. The above examples show that the diagram system cannot perform the representational

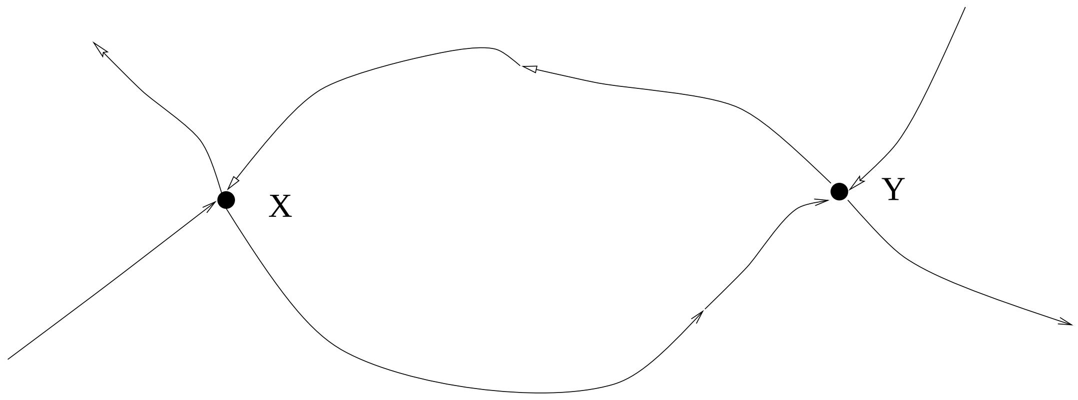

Figure 7:

task set for it. Thus, using the proposed representation system, or indeed slight generalizations of the proposal (employing algebraic curves in place of straight line segments), would lead to mistakes in logical inferences.

The study of the system LD and its variants illustrates a general point about the representational use of spatial relations; that use of such relations is only appropriate in the representation of similarly constrained structures (e.g. trivially, spatial objects and relations). The use of space in representations of more abstract structures, such as sets or models (e.g. Hammer [9, 10]), is thus to be approached with some caution.

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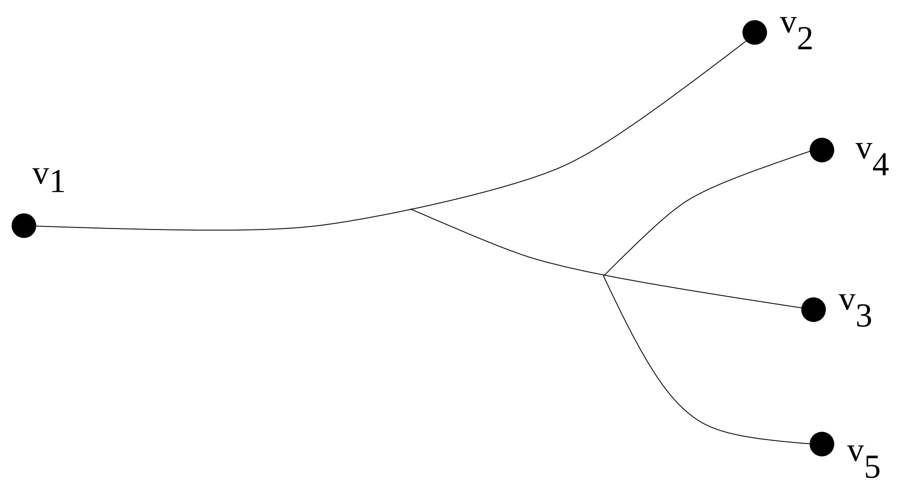

Figure 8:

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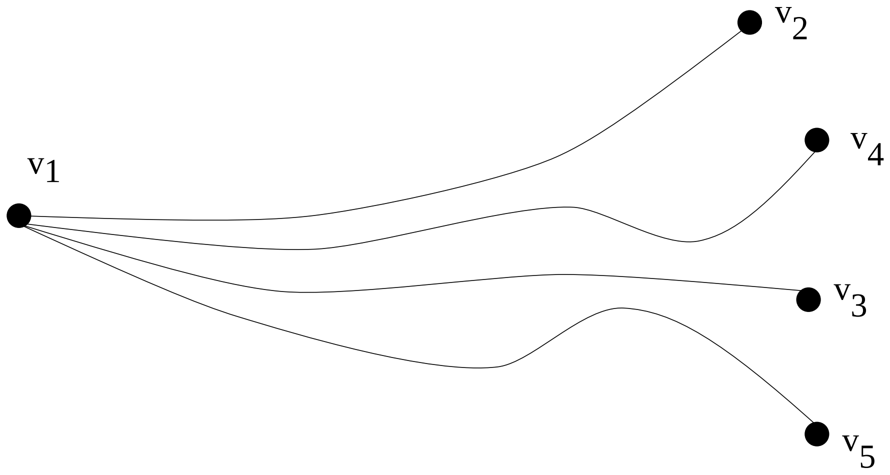

Figure 9:

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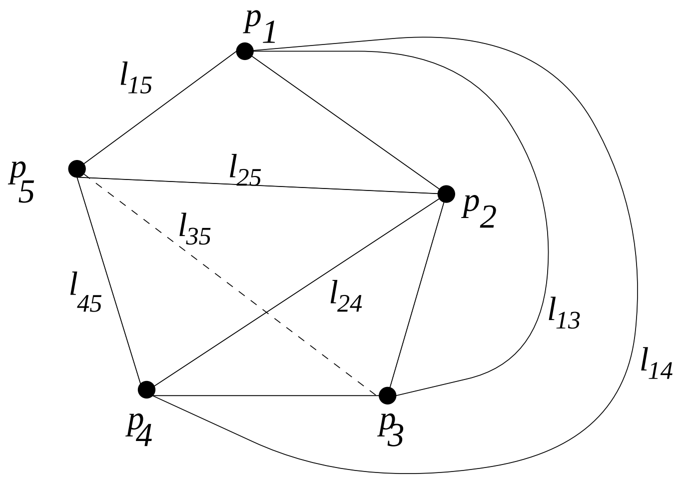

Figure 10: A counterexample using a non-planar graph (according to the premises lines l_{35} and l_{24} should not cross, but due to the system "curved LD" l_{35} must cross at least one other line if it is to join p_3 and p_5).

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