The Monoid of Inverse Maps

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Abstract

This report introduces the inverse map monoid. The monoid was found by exploring the space of all inverted maps. This monoid will build inverted maps which are bundles. The monoid is isomorphic to the monoid of maps. A number of morphisms of the inverse map monoid are introduced.

1 Introduction

This report develops an algebraic structure, the monoid of inverse maps, which will be useful in the Irish School of Constructive Mathematics, see Mac an Airchinnigh [2, 3, 4]. In which algebraic structures and morphisms are used to specify and develop software systems. This algebraic structure is an example of the many algebraic structure which under pin computer systems. The finding of new algebraic structures also expands general mathematical knowledge, and so will useful to many of the current and future computer systems.

Inverse maps are use in many specifications of software systems, such as consistently updating aliases in a system, see Mac an Airchinnigh [3, pages 43 – 45], they are also use to find the basic parts in a bill of materials, see Mac an Airchinnigh [3, page 56]. In mathematics inverse functions are used all the time, for example in the topological definition of a continuous function, see Royden [9, page 173]. Inverse functions are also used in the definition of a powerset functor, see Barr and Wells [5, pages 58 – 59].

2 Inverse Map Monoid

Let $\mathcal{M}$ denote the space of maps $X \to Y$, then the space of inverted maps, denoted $\mathcal{M}^{-1}$, is formed by taking all maps μ in $\mathcal{M}$ and finding their inverse image μ^{-1}. The space of inverted maps $\mathcal{M}^{-1}$ is a strict subspace of the space

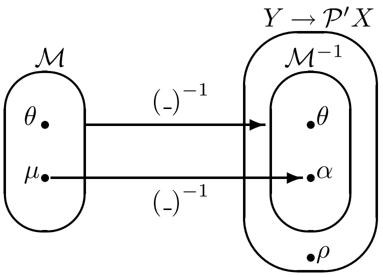

Figure 1: The spaces $\mathcal{M}$,$\mathcal{M}^{-1}$ and $Y \rightarrow \mathcal{P}'X$

$Y \rightarrow \mathcal{P}'X$ as there exits ρ in $Y \rightarrow \mathcal{P}'X$ such that $\rho \neq \mu^{-1}$ for all μ in $\mathcal{M}$, there are relations which are not inverted functions. These ρ's are systems which are not partitioned, see Hughes [7] also Hughes and Donnely [8] for more comments on partition. If α is in $\mathcal{M}^{-1}$ then $\alpha = \mu^{-1}$ for some map μ in $\mathcal{M}$. The inverse image of μ partitions the domain of μ so α is a partitioned system, α is in fact a bundle, see Goldblatt [6, pages 88 – 96].These spaces are shown in Figure 1.

Two maps μ and ν in $\mathcal{M}$ can be combined using map override to form a new map $\mu \dagger \nu$ in $\mathcal{M}$. We are naturally lead to ask, can two inverted maps α and β in $\mathcal{M}^{-1}$ be combined using some binary operation to form a new inverted map $\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta$ in $\mathcal{M}^{-1}$? The required binary operation, which is called inverse override, was obtained from an inverse image theorem, see Mac an Airchinnigh [4, page 38].

Lemma 1 *If α and β are inverted maps in $\mathcal{M}^{-1}$ then $\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta$ is an inverted map in $\mathcal{M}^{-1}$ where:*

$$\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta = (\mathcal{I} \rightarrow\!\lhd [\![{}^{\cup}/\texttt{rng}\,\beta]\!])'\alpha \circledcup \beta \tag{1}$$

where the prime denotes removal of entries of the form $y \mapsto \emptyset$.

Note the use of an indexed operator $\circledcup$ in the definition, for more on indexed operators see Mac an Airchinnigh [4, page 28 – 29] also see Donnelly, Gallagher and Hughes [1]. **Proof.** As α and β are inverted maps in $\mathcal{M}^{-1}$ there exist maps μ and ν in $\mathcal{M}$ such that $\alpha = \mu^{-1}$ and $\beta = \nu^{-1}$. If we can show that $\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta = (\mu \dagger \nu)^{-1}$ then $\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta$ is an inverted map in $\mathcal{M}^{-1}$ because $\mu \dagger \nu$ is a map in $\mathcal{M}$. Figure 2 shows these relationships.

The first step in the proof is to use the definition of the inverse override operator and then as α and β are inverted maps so $\alpha = \mu^{-1}$ and $\beta = \nu^{-1}$ where μ and ν are maps in $\mathcal{M}$ as was noted above. Also if β is an inverted map in $\mathcal{M}^{-1}$ and $\beta = \nu^{-1}$ for some map ν in $\mathcal{M}$ then ${}^{\cup}/\texttt{rng}\,\beta = \texttt{dom}\,\nu$, we will return to this relations ship later. Given the above facts we find that:

$$\begin{aligned} \alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta &= (\mathcal{I} \rightarrow\!\lhd [\![{}^{\cup}/\texttt{rng}\,\beta]\!])'\alpha \circledcup \beta \\ &= (\mathcal{I} \rightarrow\!\lhd [\![\texttt{dom}\,\nu]\!])'\mu^{-1} \circledcup \nu^{-1} \end{aligned}$$

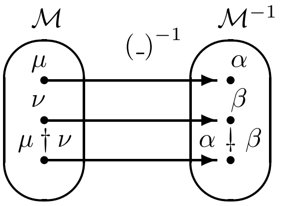

Figure 2: Closure of inverse override

We next use the identity $(\mathcal{I} \rightarrow \ntriangleleft [\![S]\!])' \mu^{-1} = (\ntriangleleft [\![S]\!] \mu)^{-1}$ where S is a set in $\mathcal{P}X$ and μ is a map in $\mathcal{M}$ followed by an application of an inverse image theorem, see Mac an Airchinnigh [4, page 37], finally, using the definition of map override in terms of extend we find that:

$$
\begin{aligned}
&= (\ntriangleleft [\![\mathtt{dom}\,\nu]\!] \mu)^{-1} \circledcirc \nu^{-1} \\
&= (\ntriangleleft [\![\mathtt{dom}\,\nu]\!] \mu \sqcup \nu)^{-1} \\
&= (\mu \dagger \nu)^{-1}
\end{aligned}
$$

We have now arrived at the required result which shows that we have a binary operation on the space of inverted maps. ∎

This proof is the reverse of a proof of an inverse image theorem, see Mac an Airchinnigh [4, pages 37 – 39]. Now that we have a binary operator on the space of inverted maps what algebraic laws does it satisfy?

Lemma 2 *Inverse override is an associative operator, that is if* α, β *and* γ *are inverted maps in* $\mathcal{M}^{-1}$ *then*

$$\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma) = (\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma \tag{2}$$

Before we proceed to prove this we must note a relationship between the space of inverted maps and the space of sets. If α and β are inverted maps in $\mathcal{M}^{-1}$ then we have the equality:

$${}^{\cup}/\mathtt{rng}\,(\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) = {}^{\cup}/\mathtt{rng}\,\alpha \cup {}^{\cup}/\mathtt{rng}\,\beta \tag{3}$$

Note that ${}^{\cup}/\mathtt{rng}$ must not be separated or else this distributive property will not hold. The composition function will distribute but the component function range will not distribute. This has implications for the development of a monoid of partitioned sets. This equality may be established be noting that $\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta = (\mu \dagger \nu)^{-1}$ where $\alpha = \mu^{-1}$ and $\beta = \nu^{-1}$ for some maps μ and ν in $\mathcal{M}$ and also noting if β is an inverted map in $\mathcal{M}^{-1}$ and $\beta = \mu^{-1}$ for some map μ in $\mathcal{M}$ then ${}^{\cup}/\mathtt{rng}\,\beta = \mathtt{dom}\,\mu$. Both of these facts were used in the proof of the first lemma. Finally the $\mathtt{dom}$ epimorphism is also used:

$$
\begin{aligned}
{}^{\cup}/\mathtt{rng}\,(\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) &= \mathtt{dom}\,(\mu \dagger \nu) \\
&= \mathtt{dom}\,\mu \cup \mathtt{dom}\,\nu \\
&= {}^{\cup}/\mathtt{rng}\,\alpha \cup {}^{\cup}/\mathtt{rng}\,\beta
\end{aligned}
$$

Hence our equality is proven, we will return to this later and talk about its algebraic meaning. So we proceed by proving the above lemma, the associative law for the inverse override operator.

Proof. The definition of the inverse override operator is first use in the proof followed by an application of the above equality and then we use the commutativity of set union followed by the composition law of removal endomorphisms, see Mac an Airchinnigh [2, pages 125 – 128]. The choice to start from *alpha* $\mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma)$ is important as the proof turns out to be simpler, this is because the operator is non commutative.

$$\begin{aligned}
\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma) &= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,(\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma)\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma) \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\beta \cup {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma) \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma \cup {}^{\cup}/\mathtt{rng}\,\beta\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma) \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket \circ \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\beta\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma)
\end{aligned}$$

Next we apply the composition law of removal functors and then we use the definition of the inverse override operator and finally using the fact that the removal functor is a homomorphism we find that:

$$\begin{aligned}
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket)'(\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\beta\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} (\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma) \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket)'(\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\beta\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket)'\beta \mathbin{\bigcirc\hspace{-0.72em}\cup} \gamma \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket)'((\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\beta\rrbracket)'\alpha \mathbin{\bigcirc\hspace{-0.72em}\cup} \beta) \mathbin{\bigcirc\hspace{-0.72em}\cup} \gamma \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\gamma\rrbracket)'(\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) \mathbin{\bigcirc\hspace{-0.72em}\cup} \gamma \\
&= (\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \gamma
\end{aligned}$$

So the inverse override operator is associative. ∎

The next obvious question to ask is about the existence of an identity element for the inverse override operator.

Lemma 3 *The null map is the identity element for inverse override, that is if β is an inverted map in $\mathcal{M}^{-1}$ and θ is the null map then*

$$\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \theta = \beta = \theta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta \tag{4}$$

Proof. The proof of this lemma is just an application of the definition of the inverse override operator:

$$\begin{aligned}
\beta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \theta &= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\theta\rrbracket)'\beta \mathbin{\bigcirc\hspace{-0.72em}\cup} \theta \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket \emptyset \rrbracket)'\beta \\
&= \beta \\
&= \theta \mathbin{\bigcirc\hspace{-0.72em}\cup} \beta \\
&= (\mathcal{I} \to \lhd\!\!\!- \llbracket {}^{\cup}/\mathtt{rng}\,\beta\rrbracket)'\theta \mathbin{\bigcirc\hspace{-0.72em}\cup} \beta \\
&= \theta \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta
\end{aligned}$$

So the null map is the identity element for inverse override. ∎

The above three lemmas are summed up by the following theorem on the algebraic structure of the space of inverted maps.

Theorem 1 (Inverse Map Monoid) *If $\mathcal{M}^{-1}$ is the space of inverted maps obtained form the space of maps $\mathcal{M}$ then $(\mathcal{M}^{-1}, \mathbin{\rotatebox[origin=c]{180}{$\dagger$}}, \theta)$ is a monoid where*

$$\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta = (\mathcal{I} \rightarrow\!\lhd [\![{}^{\cup}/\mathtt{rng}\,\beta]\!])'\alpha \mathbin{\textcircled{$\cup$}} \beta \tag{5}$$

for inverted maps α and β in $\mathcal{M}^{-1}$.

We now return to the algebraic meaning of the equality, ${}^{\cup}/\mathtt{rng}\,\mu^{-1} = \mathtt{dom}\,\mu$, introduced earlier. Now that the space of inverted maps has a structure, that of a monoid, we have the following lemma.

Lemma 4 *The function ${}^{\cup}/\mathtt{rng} : \mathcal{M}^{-1} \rightarrow \mathcal{P}X$ is an epimorphism from the monoid of inverted maps $(\mathcal{M}^{-1}, \mathbin{\rotatebox[origin=c]{180}{$\dagger$}}, \theta)$ to the monoid of sets $(\mathcal{P}X, \cup, \emptyset)$.*

Proof. The function is onto as ${}^{\cup}/\mathtt{rng}\,(\mathcal{M}^{-1}) = \mathcal{P}X$. Next we want to show that the function is a homomorphism, that is if α and β are inverted maps in $\mathcal{M}^{-1}$ then:

$${}^{\cup}/\mathtt{rng}\,(\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) = {}^{\cup}/\mathtt{rng}\,\alpha \cup {}^{\cup}/\mathtt{rng}\,\beta \tag{6}$$

This was shown when the equality was first introduced, and so the function is an epimorphism. ∎

This epimorphism is of course just the $\mathtt{dom}$ epimorphism in an inverted world. The existence of this epimorphism hints at how closely the monoid of maps is related to the monoid of inverted maps, as shown in the theorem below.

Theorem 2 *The monoid of maps $(\mathcal{M}, \dagger, \theta)$ is isomorphic to the monoid of inverted maps $(\mathcal{M}^{-1}, \mathbin{\rotatebox[origin=c]{180}{$\dagger$}}, \theta)$.*

Proof. The isomorphism is the inverse image function $(\_)^{-1} : \mathcal{M} \rightarrow \mathcal{M}^{-1}$. Inverse image is one to one because if μ and ν are maps in $\mathcal{M}$ and if $(\mu)^{-1} = (\nu)^{-1}$ then $\mu = \nu$. Also inverse image is onto by the definition of the space $\mathcal{M}^{-1}$. Finally we want to show that inverse image is a homomorphism, that is if μ and ν are maps in $\mathcal{M}$ then:

$$(\mu \dagger \nu)^{-1} = \mu^{-1} \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \nu^{-1} \tag{7}$$

This was shown in the proof of the first lemma and so inverse image is an isomorphism. ∎

As the monoid of maps has morphisms associated with it, the monoid of inverted maps should have corresponding morphisms because the monoids are isomorphic. The epimorphism $\mathtt{dom}$ of the map monoid corresponds with the ${}^{\cup}/\mathtt{rng}$ epimorphism of the inverse map monoid. Set removal is an endomorphism of the map monoid, so what is the corresponding endomorphism of the inverse map monoid? The corresponding function is $(\mathcal{I} \rightarrow\!\lhd [\![S]\!])' : \mathcal{M}^{-1} \rightarrow \mathcal{M}^{-1}$ and it is shown to be a endomorphism in the lemma below:

Lemma 5 *The function $(\mathcal{I} \rightarrow\!\lhd [\![S]\!])' : \mathcal{M}^{-1} \rightarrow \mathcal{M}^{-1}$ where S is a set in $\mathcal{P}X$ is an endomorphism of the monoid of inverted maps $(\mathcal{M}^{-1}, \mathbin{\rotatebox[origin=c]{180}{$\dagger$}}, \theta)$.*

Proof. If S is a set in $\mathcal{P}X$ then we want to show that the function $(\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])' : \mathcal{M}^{-1} \rightarrow \mathcal{M}^{-1}$ is a homomorphism that is if α and β are inverted maps in $\mathcal{M}^{-1}$ then:

$$(\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'(\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) = (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\beta \qquad (8)$$

To show this we again note that $\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta = (\mu \dagger \nu)^{-1}$ where $\alpha = \mu^{-1}$ and $\beta = \nu^{-1}$ for some maps μ and ν in $\mathcal{M}$ and we also make use of an identity which was used in the proof of first lemma, $(\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\mu^{-1} = (\ntriangleleft[\![S]\!]\mu)^{-1}$ where S is a set in $\mathcal{P}X$ and μ is a map in $\mathcal{M}$, then the fact that set removal is an endomorphism of the map monoid is used:

$$\begin{array}{rcl} (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'(\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} \beta) & = & (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'(\mu \dagger \nu)^{-1} \\ & = & (\ntriangleleft[\![S]\!](\mu \dagger \nu))^{-1} \\ & = & (\ntriangleleft[\![S]\!]\mu \dagger \ntriangleleft[\![S]\!]\nu)^{-1} \end{array}$$

Next we use the fact that the map monoid is isomorphic to the inverse map monoid, by inverse image, and finally another application of the identity used above:

$$\begin{array}{rl} = & (\ntriangleleft[\![S]\!]\mu)^{-1} \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\ntriangleleft[\![S]\!]\nu)^{-1} \\ = & (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\mu^{-1} \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\nu^{-1} \\ = & (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\alpha \mathbin{\rotatebox[origin=c]{180}{$\dagger$}} (\mathcal{I}\rightarrow\ntriangleleft[\![S]\!])'\beta \end{array}$$

This shows that the function is an endomorphism. ▮

The morphism corresponding to the set restriction endomorphism of the map monoid is $(\mathcal{I} \rightarrow \triangleleft[\![S]\!])' : \mathcal{M}^{-1} \rightarrow \mathcal{M}^{-1}$. The space of operators $(\mathcal{I} \rightarrow\ntriangleleft[\![\mathcal{P}X]\!])'$ forms a monoid under functional composition as does the space of operators $(\mathcal{I} \rightarrow \triangleleft[\![\mathcal{P}X]\!])'$. These two monoids are isomorphic. The details of the above statements are left as an exercise to the interested reader.

3 Summary

An inverse map monoid was found and a number of morphisms of this monoid were examined, one was found to be an isomorphism with the map monoid. Finally an alternative inverse map monoid was hinted at.

I wish to thank Dr. Mícheál Mac an Airchinnigh for starting this debate in his previous works. I also owe a debt of gratitude to the other members of the the Irish School of Constructive Mathematics, Dr. Andrew Butterfield, Dr. Hugh Gibbons, Alexis Donnelly, John Walsh, Andrew Farrell, Colman Reilly and Dara Gallagher, for their invaluable suggestions and critical review of this report.

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